

习题 4.4

$$1. (1) e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, (-\infty < x < \infty);$$

$$(2) \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{3^{n+1}(n+1)}, (-3 < x < 3);$$

$$(3) \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{2n} \quad (-\sqrt{2} < x < \sqrt{2})$$

$$2. (1) \sum_{n=0}^{\infty} \frac{(\ln a)^n x^n}{n!}, x \in (-\infty, +\infty); (2) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} x^{2n+1}, x \in (-\infty, +\infty);$$

$$(3) \sum_{n=0}^{\infty} \frac{x^n}{a^{n+1}}, x \in (-|a|, |a|); (4) \ln a + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{na^n} x^n, x \in (-a, a];$$

$$(5) \frac{3}{4} \sum_{n=1}^{\infty} \frac{(-1)^n (1-3^{2n})}{(2n+1)!} x^{2n+1}, x \in (-\infty, +\infty);$$

$$(6) \frac{\sqrt{2}}{2} \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \right), x \in (-\infty, +\infty);$$

$$(7) \sum_{n=1}^{\infty} \left(\frac{(-1)^{n-1} 2^n - 1}{n} \right) x^n, x \in \left(-\frac{1}{2}, \frac{1}{2} \right];$$

$$(8) \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2^n + 1}{3} x^n, x \in \left(-\frac{1}{2}, \frac{1}{2} \right);$$

$$(9) \sum_{n=0}^{\infty} \frac{x^{4n+1}}{4n+1}, x \in (-1, 1)$$

$$3. (1) (x-1)^2 + 4(x-1) + 4, x \in (-\infty, +\infty);$$

$$(2) \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(x + \frac{\pi}{3} \right)^{2n} + \frac{\sqrt{3}}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(x + \frac{\pi}{3} \right)^{2n+1}, x \in (-\infty, +\infty);$$

$$(3) \sum_{n=0}^{\infty} \frac{e(x-1)^n}{n!}, x \in (-\infty, +\infty);$$

$$(4) \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{n+1}} (x-3)^n, x \in (0, 6)$$

4. 将下列函数展开成 x 的幂级数:

$$(1) x + \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n+1)(2n)!!} x^{2n+1}, x \in [-1, 1];$$

$$(2) \frac{1}{(2-x)^2} = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{2^{n+1}}, x \in (-2, 2).$$

$$5. (1) \operatorname{sh} x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, x \in (-\infty, +\infty).$$

$$(2) \sin^2 x = \frac{1}{2} - \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n-1} x^{2n}}{(2n)!}, x \in (-\infty, +\infty).$$

$$(3) (1+x) \ln(1+x) = x + \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)} x^n, x \in (-1, 1].$$

$$(4) \frac{x}{\sqrt{1+x^2}} = x + \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} x^{2n+1}, x \in [-1, 1].$$

$$6. (1) \sqrt{x^3} = 1 + \frac{3}{2}(x-1) + \sum_{n=2}^{\infty} \frac{(-1)^n 3(2n-5)!!}{2^n n!} (x-1)^n, x \in [0, 2].$$

$$(2) \lg x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \ln 10} (x-1)^n, x \in (0, 2].$$

$$7. \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{\left(x + \frac{\pi}{3}\right)^{2n}}{(2n)!} + \frac{\sqrt{3}}{2} \sum_{n=0}^{\infty} (-1)^n \frac{\left(x + \frac{\pi}{3}\right)^{2n+1}}{(2n+1)!}, x \in (-\infty, +\infty).$$

$$8. \sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{3^{n+1}}, x \in (0, 6).$$

$$9. \sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}} \right) (x+4)^n, x \in (-6, -2)$$

$$10. \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{4^{n+1}} - \frac{1}{5^{n+1}} \right) (x-2)^n (|x-2| < 4)$$

$$11. \frac{1}{x^2} = \sum_{n=1}^{\infty} \frac{n}{4^{n+1}} (x+4)^{n-1} (|x+4| < 4).$$

$$12. f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} x^{2n+1}, x \in (-\infty, +\infty).$$

$$13. f(x) = \cos x = \cos x_0 + \cos\left(x_0 + \frac{\pi}{2}\right)(x-x_0) + \cdots + \frac{\cos\left(x_0 + \frac{n\pi}{2}\right)}{n!} (x-x_0)^n + \cdots.$$

$$14. (1) 2e; (2) \frac{\cos 1 + \sin 1}{2}.$$

$$15. \ln 2 \approx 0.6931.$$

$$16. \sin 9^\circ \approx 0.15643, \text{ 误差不超过 } 10^{-5}.$$

$$17. e^x \cos x = \sum_{n=0}^{\infty} \frac{2^{\frac{n}{2}} \cos \frac{n\pi}{4}}{n!} x^n;$$

$$18. \sum_{n=1}^{\infty} \frac{nx^{n-1}}{(n+1)!}.$$