习题 5. 3

1. $E(\overline{X})=\lambda$, $D(\overline{X})=\dfrac{\lambda}{n}$, $E(S^2)=\lambda$.

2. (1)0. 131 4, (2)0. 177 6.

解：(1)$\because \overline{X}\sim N\left(12,\dfrac{4}{5}\right)$,

$$\therefore P(\overline{X}>13)=P\left(\frac{\overline{X}-12}{2/\sqrt{5}}>\frac{13-12}{2/\sqrt{5}}\right)=P\left(\frac{\overline{X}-12}{2/\sqrt{5}}>\frac{\sqrt{5}}{2}\right)$$

$=1-\Phi(1.12)=1-0.8686=0.1314$

(2)$P\{X_{(1)}>10\}=P\{X_1>10,X_2>10,\cdots,X_{10}>10\}=[P\{X>10\}]^{10}$

$=[1-P\{X\leqslant 10\}]^{10}=[1-\Phi(-1)]^{10}=[\Phi(1)]^{10}=(0.8413)^{10}\approx 0.1776$

3. (1) $\chi^2(n-1)$; (2)$t(n-1)$; (3) $\chi^2(2)$.

解：(1) $\dfrac{nS_n^2}{\sigma^2}=\dfrac{n}{\sigma^2}\cdot\dfrac{n-1}{n}S^2=\dfrac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1)$

(2)$\because \dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}}\sim N(0,1)$, $\dfrac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1)$

$$\therefore \frac{\overline{X}-\mu}{\dfrac{S_n}{\sqrt{n-1}}}=\frac{\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\dfrac{(n-1)S^2}{\sigma^2}/n-1}}\sim t(n-1)$$

(3)$\because \dfrac{X_i-\mu}{\sigma}\sim N(0,1)$

$$\therefore \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2=\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2\sim\chi^2(n)$$

4. 0. 133 6.

5. 0. 05.

解：$\because X\sim N(0,0.4^2)$, $\therefore \dfrac{X}{0.4}\sim N(0,1)$

$$P\left\{\sum_{i=1}^{15}X_i^2>3.999\right\}=P\left\{\sum_{i=1}^{15}\frac{X_i^2}{0.4^2}>\frac{3.999}{0.4^2}\right\}=P\{\chi^2(15)>24.99\}=0.05$$

6. 0. 05.

解：$P\{S^2>5.6\}=P\left\{\dfrac{(n-1)S^2}{\sigma^2}>\dfrac{(n-1)5.6}{\sigma^2}\right\}=P\{\chi^2(39)>54.6\}=0.05$

习题五

1. (1) $P\{X_1=x_1, X_2=x_2, \cdots, X_n=x_n\} = \dfrac{\lambda^{\sum_{i=1}^{n}x_i}}{x_1!\ x_2!\ \cdots,\ x_n!}e^{-n\lambda}$;

(2) $P\{X_1=x_1, X_2=x_2, \cdots, X_n=x_n\} = p^{\sum_{i=1}^{n}x_i}(1-p)^{n^2-\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}C_n^{x_i}$.

解: (1) 因为总体 X 服从参数为 λ 的泊松分布, 则

$$P\{X_i=x_i\} = \frac{\lambda^{x_i}}{x_i!}e^{-\lambda}, \ (i=1, 2, \cdots, n)$$

所以

$$P\{X_1=x_1, X_2=x_2, \cdots, X_n=x_n\} = \prod_{i=1}^{n}P\{X_i=x_i\} = \frac{\lambda^{\sum_{i=1}^{n}x_i}}{x_1!\ x_2!\ \cdots x_n!}e^{-n\lambda},$$

(2) 因为总体 $X \sim B(n, p)$ 则

$$P\{X_i=x_i\} = C_n^{x_i}p^{x_i}(1-p)^{1-x_i}, \ (i=1, 2, \cdots, n)$$

所以

$$P\{X_1=x_1, X_2=x_2, \cdots, X_n=x_n\} = \prod_{i=1}^{n}P\{X_i=x_i\} = p^{\sum_{i=1}^{n}x_i}(1-p)^{n-\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}C_n^{x_i}.$$

2. $F(1, n)$; 3. $t(n-1)$;

4. $t(m+n-2)$;

解: $\because \alpha(\overline{X}-a)+\beta(\overline{Y}-b) \sim N\left(0, \left(\dfrac{\alpha^2}{m}+\dfrac{\beta^2}{n}\right)\sigma^2\right)$,

$$\therefore \frac{\alpha(\overline{X}-a)+\beta(\overline{Y}-b)}{\sigma\sqrt{\dfrac{\alpha^2}{m}+\dfrac{\beta^2}{n}}} \sim N(0, 1).$$

又 $\dfrac{(m-1)S_1^2}{\sigma^2} \sim \chi^2(m-1)$, $\dfrac{(n-1)S_2^2}{\sigma^2} \sim \chi^2(n-1)$, S_1^2, S_2^2 独立, 由 χ^2 分布的可加性知

$\dfrac{(m-1)S_1^2}{\sigma^2}+\dfrac{(n-1)S_2^2}{\sigma^2} = \dfrac{(m-1)S_1^2+(n-1)S_2^2}{\sigma^2} \sim \chi^2(m+n-2)$, 所以

$$\frac{\dfrac{\alpha(\overline{X}-a)+\beta(\overline{Y}-b)}{\sigma\sqrt{\dfrac{\alpha^2}{m}+\dfrac{\beta^2}{n}}}}{\sqrt{\dfrac{(m-1)S_1^2+(n-1)S_2^2}{\sigma^2}\Big/ m+n-2}}=\frac{\alpha(\overline{X}-a)+\beta(\overline{Y}-b)}{\sqrt{\dfrac{(m-1)S_1^2+(n-1)S_2^2}{m+n-2}}\sqrt{\dfrac{\alpha^2}{m}+\dfrac{\beta^2}{n}}}\sim t(m+n-2)$$

5. 0. 9；

解： $\because X_1+X_2\sim N(0, 2\sigma^2)$，$\therefore \dfrac{X_1+X_2}{\sqrt{2}\sigma}\sim N(0, 1)$，从而$\left(\dfrac{X_1+X_2}{\sqrt{2}\sigma}\right)^2\sim\chi^2(1)$；

$\because X_3-X_4\sim N(0, 2\sigma^2)$，$\therefore \dfrac{X_3-X_4}{\sqrt{2}\sigma}\sim N(0, 1)$，从而$\left(\dfrac{X_3-X_4}{\sqrt{2}\sigma}\right)^2\sim\chi^2(1)$，

由 F 分布的定义知$\dfrac{(X_1+X_2)^2}{(X_3-X_4)^2}=\dfrac{\left(\dfrac{X_1+X_2}{\sqrt{2}\sigma}\right)^2\Big/1}{\left(\dfrac{X_3-X_4}{\sqrt{2}\sigma}\right)^2\Big/1}\sim F(1, 1)$，

所以 $P\left\{\dfrac{(X_1+X_2)^2}{(X_3-X_4)^2}<40\right\}=1-P\left\{\dfrac{(X_1+X_2)^2}{(X_3-X_4)^2}\geqslant 40\right\}=1-0.1=0.9.$

6. $k=4.265$；

7. $F(5, n-5)$.

8. 35.　9. (1)40，(2)1537.

10. (1)0. 94，(2)0. 895.　11. 0. 829 3.　12. 0. 133 6.

13. 0. 8104.　14. 0. 66.

解： 设 $n_1=20$，$n_2=25$，

$\because X\sim N(30, 3^2)$，$Y\sim N(30, 3^2)$ $\therefore \dfrac{\overline{X}-\overline{Y}}{3\sqrt{\dfrac{1}{20}+\dfrac{1}{25}}}\sim N(0, 1)$，

$$\therefore P\{|\overline{X}-\overline{Y}|>0.4\}=P\left\{\left|\frac{\overline{X}-\overline{Y}}{3\sqrt{\dfrac{1}{20}+\dfrac{1}{25}}}\right|>\frac{0.4}{3\sqrt{\dfrac{1}{20}+\dfrac{1}{25}}}\right\}$$

$=2-2\Phi(0.44)=2-2\times 0.67=0.66.$

15. (1)0. 262 8；(2)0. 292 3，0. 578 5.

解： (1) $\because X\sim N(12, 4)$，$n=5$，$\therefore \dfrac{\overline{X}-12}{2/\sqrt{5}}\sim N(0, 1)$，

$$\therefore P\{|\overline{X}-12|>1\}=P\left\{\left|\frac{\overline{X}-12}{2/\sqrt{5}}\right|>\frac{1}{2/\sqrt{5}}\right\}=P\left\{\left|\frac{\overline{X}-12}{2/\sqrt{5}}\right|>1.12\right\}$$

$= 2 - 2\Phi(1.12) = 2 - 2\times 0.8686 = 0.2628.$

(2) $P\{\max(X_1, X_2, \cdots, X_5) > 15\} = 1 - P\{\max(X_1, X_2, \cdots, X_5) \leqslant 15\}$

$= 1 - P\{X_1 \leqslant 15, X_2 \leqslant 15, \cdots, X_5 \leqslant 15\} = 1 - [P\{X \leqslant 15\}]^5$

$= 1 - [\Phi(1.5)]^5 = 1 - [0.9332]^5 = 0.2923$

$P\{\min(X_1, X_2, \cdots, X_5) < 10\} = 1 - P\{\min(X_1, X_2, \cdots, X_5) \geqslant 10\}$

$= 1 - P\{X_1 \geqslant 10, X_2 \geqslant 10, \cdots, X_5 \geqslant 10\} = 1 - [P\{X \geqslant 10\}]^5$

$= 1 - [\Phi(1)]^5 = 1 - [0.8413]^5 = 0.5786$

16. $2(n-1)\sigma^2$.

17. (1)0.99；(2) $\dfrac{2\sigma^4}{n-1}$.

解：(1) $P\left\{\dfrac{S^2}{\sigma^2} \leqslant 2.041\right\} = P\left\{\dfrac{(n-1)S^2}{\sigma^2} \leqslant 2.041\times(n-1)\right\}$

$= P\left\{\dfrac{(n-1)S^2}{\sigma^2} \leqslant 30.615\right\} = P\{\chi^2(n-1) \leqslant 30.615\}$

$= 1 - P\{\chi^2(n-1) > 30.615\} = 1 - 0.01 = 0.99$

(2) $\because \dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$，$\therefore D\left[\dfrac{(n-1)S^2}{\sigma^2}\right] = 2(n-1)$，

由 $D\left[\dfrac{(n-1)S^2}{\sigma^2}\right] = \dfrac{(n-1)^2}{\sigma^4}D(S^2) = 2(n-1)$，得 $D(S^2) = \dfrac{2\sigma^4}{n-1}$

18. (1) $f_Y(y) = f\left(\dfrac{y}{2\lambda}\right)\dfrac{1}{2\lambda} = \begin{cases} \dfrac{1}{2}e^{-\frac{y}{2}}; & y > 0 \\ 0, & y \leqslant 0 \end{cases}$；(2) $2n\lambda\overline{X} \sim \chi^2(2n)$.

解：(1) 由公式法可知，$f_Y(y) = f\left(\dfrac{y}{2\lambda}\right)\dfrac{1}{2\lambda} = \begin{cases} \dfrac{1}{2}e^{-\frac{y}{2}}, & y > 0 \\ 0, & y \leqslant 0 \end{cases}$

将 Y 的概率密度对比 χ^2 分布的概率密度的表达式可知，$Y \sim \chi^2(2)$，

(2) $2n\lambda\overline{X} = 2\lambda(X_1 + X_2 + \cdots + X_n)$，由 χ^2 分布的可加性可知 $2n\lambda\overline{X} \sim \chi^2(2n)$

19. $\dfrac{2}{n(n-1)}$.

解：当 $\mu = 0$，$\sigma = 1$ 时，$\overline{X} \sim N\left(0, \dfrac{1}{n}\right)$

$$\frac{\overline{X} - 0}{\sqrt{\frac{1}{n}}} \sim N(0, 1)\text{，则}\left(\frac{\overline{X} - 0}{\sqrt{\frac{1}{n}}}\right)^2 = n\overline{X}^2 \sim \chi^2(1)$$

利用若 $Y \sim \chi^2(n)$，则 $D(Y) = 2n$，得

$D(n\overline{X}^2)=2$，即 $D(\overline{X}^2)=\dfrac{2}{n^2}$

而 $\overline{X}$ 和 S^2 独立，则 $D(T)=D(\overline{X}^2-\dfrac{1}{n}S^2)=D(\overline{X}^2)+\dfrac{1}{n^2}D(S^2)$

而由 17 题知，$D(S^2)=\dfrac{2\sigma^4}{n-1}$，而本题中 $\sigma=1$，所以 $D(S^2)=\dfrac{2}{n-1}$

综上，得 $D(T)=\dfrac{1}{n^2}\cdot 2+\dfrac{1}{n^2}\cdot\dfrac{2}{n-1}=\dfrac{2}{n(n-1)}$.