习题 6.2

1. (1) $\hat{\theta} = 2\overline{X} - \frac{1}{2}$; (2) 因为 $E(4\overline{X}^2) > \theta^2$, 所以 $4\overline{X}^2$ 不是 θ^2 的无偏估计.

2. **解**: $X \sim E(\theta)$, $E(\overline{X}) = E(X) = \theta$, 所以 $\overline{X}$ 是 θ 的无偏估计.

令 $T = \min\{X_1, X_2, \cdots, X_n\}$,

$$\begin{aligned}
T \text{ 的分布函数 } F_T(t) &= P(T \leqslant t) \\
&= 1 - P(T > t) \\
&= 1 - P(X_1 > t, X_2 > t, \cdots, X_n > t) \\
&= 1 - [P(X > t)]n \\
&= 1 - [1 - P(X \leqslant t)]n \\
&= 1 - [1 - F_X(t)]n \\
&= 1 - e^{-\frac{nt}{\theta}}, \ t > 0,
\end{aligned}$$

T 的概率密度 $f_T(t) = (F_T(t))' = \frac{n}{\theta}e^{-\frac{nt}{\theta}}, \ t > 0$,

即 $T \sim E\left(\frac{\theta}{n}\right)$, 故 $E(T) = \frac{\theta}{n}$, 从而 $E(nT) = \theta$,

所以 nT 也是 θ 的无偏估计.

3. μ_2 最有效.

4. $\frac{4}{3}\max\{X_1, X_2, X_3\}$ 比 $4\min\{X_1, X_2, X_3\}$ 更有效.

解: 令 $M = \max\{X_1, X_2, X_3\}$, $N = \min\{X_1, X_2, X_3\}$, 先求 M, N 的概率密度.

$$F_M(x) = P\{M \leqslant x\} = P\{X_1 \leqslant x, X_2 \leqslant x, X_3 \leqslant x\}$$

$$= [P\{X \leqslant x\}]3 = \begin{cases} 1, & x \geqslant \theta, \\ \frac{x^3}{\theta^3}, & 0 < x < \theta, \\ 0, & x \leqslant 0. \end{cases}$$

$$\therefore f_M(x) = \begin{cases} \frac{3x^2}{\theta^3}, & 0 < x < \theta, \\ 0, & \text{其他}. \end{cases}$$

$$F_N(x) = P\{N \leqslant x\} = 1 - P\{N > x\} = 1 - P\{X_1 \geqslant x, X_2 \geqslant x, X_3 \geqslant x\}$$

$$= 1 - [1 - P\{X < x\}]3 = \begin{cases} 1, & x \geqslant \theta, \\ 1 - \left(1 - \frac{x}{\theta}\right)^3, & 0 < x < \theta, \\ 0, & x \leqslant 0. \end{cases}$$

$$\therefore f_N(x) = \begin{cases} \frac{3}{\theta}\left(1 - \frac{x}{\theta}\right)^2, & 0 < x < \theta, \\ 0, & \text{其他}. \end{cases}$$

$$E\left(\frac{4}{3}M\right)=\frac{4}{3}\int_0^\theta \frac{3x^3}{\theta^3}\mathrm{d}x=\int_0^\theta \frac{4x^3}{\theta^3}\mathrm{d}x=\theta,$$

$$E(4N)=4\int_0^\theta\left[\frac{3}{\theta}\left(1-\frac{x}{\theta}\right)^2\right]x\mathrm{d}x=\frac{12}{\theta}\int_0^\theta\left(x+\frac{x^3}{\theta^2}-\frac{2x^2}{\theta}\right)\mathrm{d}x=\theta,$$

$\therefore\ \frac{4}{3}\max\{X_1,X_2,X_3\}$，$4\min\{X_1,X_2,X_3\}$ 都是 θ 的无偏估计.

$$E(M)=\frac{3}{4}\theta,\ E(M^2)=\int_0^\theta \frac{3x^4}{\theta^3}\mathrm{d}x=\frac{3\theta^2}{5},$$

$$E(N)=\frac{\theta}{4},\ E(N^2)=\int_0^\theta\left[\frac{3}{\theta}\left(1-\frac{x}{\theta}\right)^2\right]x^2\mathrm{d}x=\frac{3}{\theta}\int_0^\theta\left(x^2+\frac{x^4}{\theta^2}-\frac{2x^3}{\theta}\right)\mathrm{d}x=\frac{\theta^2}{10},$$

$$\therefore D(M)=E(M^2)-E^2(M)=\frac{3}{80}\theta^2,\ D(N)=E(N^2)-E^2(N)=\frac{3}{80}\theta^2,$$

$$\therefore D\left(\frac{4}{3}M\right)=\frac{1}{15}\theta^2<D(N)=\frac{3}{5}\theta^2,$$

$\therefore\ \frac{4}{3}\max\{X_1,X_2,X_3\}$ 比 $4\min\{X_1,X_2,X_3\}$ 更有效.

5. **解**：$X\sim U(0,\theta)$，$D(X)=\sigma^2=\frac{\theta^2}{12}$,

$$\because E\left[\frac{12}{n-1}\sum_{i=1}^n(X_i-\overline{X})^2\right]=12E(S^2)=12D(X)=12\cdot\frac{\theta^2}{12}=\theta^2,$$

$\therefore\ \frac{12}{n-1}\sum_{i=1}^n(X_i-\overline{X})^2$ 是 θ^2 的无偏估计.

$$\because\ \frac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1),\ D\left[\frac{(n-1)S^2}{\sigma^2}\right]=2(n-1),$$

$$\therefore D(S^2)=\frac{2\sigma^4}{n-1}=\frac{\theta^4}{72(n-1)}.$$

由切比雪夫不等式，$P\{|12S^2-\theta^2|<\varepsilon\}\geqslant 1-\frac{D(12S^2)}{\varepsilon^2}=1-\frac{144D(S^2)}{\varepsilon^2}=1-\frac{2\theta^4}{(n-1)\varepsilon^2}$

$$\lim_{n\to\infty}P\{|12S^2-\theta^2|<\varepsilon\}\geqslant\lim_{n\to\infty}\left[1-\frac{2\theta^4}{(n-1)\varepsilon^2}\right]=1,$$

$\therefore\ \frac{12}{n-1}\sum_{i=1}^n(X_i-\overline{X})^2$ 是 θ^2 的相合估计.